BIFURCATIONS OF UMBILIC POINTS AND RELATED PRINCIPAL CYCLES

CARLOS GUTIÉRREZ, JORGE SOTOMAYOR AND RONALDO GARCIA

ABSTRACT.- The simplest patterns of qualitative changes on the *configurations of lines of principal curvature* around umbilic points on surfaces whose immersions into $\mathbb{R}^3$ depend smoothly on a real parameter (codimension one umbilic bifurcations) are described in this paper. Global effects, due to umbilic bifurcations, on these configurations such as the appearance and annihilation of periodic principal lines, called also principal cycles, are also studied here.

1. Introduction

Let $\mathbb{M}^2$ be a connected, compact, oriented, two dimensional smooth manifold and let $\mathbb{R}^3$ be the *Euclidean* $3-$ *space*, endowed with a once for all fixed *orientation*, *Euclidean inner product*: $<,>$ and *norm*: $|\ | = <,>^{1/2}$.

An immersion α of $\mathbb{M}^2$ into $\mathbb{R}^3$ is a map such that the derivative $D\alpha_p$ from $\mathbb{TM}^2_p$ to $\mathbb{R}^3$ is one to one, for every $p \in \mathbb{M}^2$. Here $\mathbb{TM}^2_p$ stands for the tangent space of $\mathbb{M}^2$ at p. The tangent bundle of $\mathbb{M}^2$ will be denoted by $\mathbb{TM}^2$. The tangent projective bundle will be denoted by $\mathbb{PM}^2$ and its projection onto $\mathbb{M}^2$ will be designated by Π. The fiber $\mathbb{PM}^2_p = \Pi^{-1}(p)$ is the projective line over p which is the space of tangent lines through p.

Denote by $\mathcal{I}^r = \mathcal{I}^r(\mathbb{M}^2, \mathbb{R}^3)$ the set of C^r-immersions of $\mathbb{M}^2$ into $\mathbb{R}^3$. When endowed with the C^s, $s \leq r$, topology this space will be denoted by $\mathcal{I}^{r,s} = \mathcal{I}^{r,s}(\mathbb{M}^2, \mathbb{R}^3)$.

To every $\alpha \in \mathcal{I}^r$ is associated its *Gaussian normal map* $N_\alpha : \mathbb{M}^2 \to \mathbb{S}^2$, with values in the unit 2-sphere $\mathbb{S}^2$, defined by

$$N_\alpha(p) = (|\alpha_u \wedge \alpha_v|^{-1})\alpha_u \wedge \alpha_v,$$

where $(u, v) : (\mathbb{M}^2, p) \to (\mathbb{R}^2, 0)$ is a local positive chart on an open set of $\mathbb{M}^2$ containing p; $\wedge$ denotes the *vector* or *wedge product* in the

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oriented space $\mathbb{R}^3$; $\alpha_u = \partial\alpha/\partial u$ and $\alpha_v = \partial\alpha/\partial v$. Clearly N_α is well defined and of class C^{r-1} in $\mathbb{M}^2$.

Since $DN_\alpha(p)$ has its image contained in the image of $D\alpha(p)$, the *Weingarten endomorphism* $\omega_\alpha : \mathbb{TM}^2 \to \mathbb{TM}^2$ is well defined by

$$(D\alpha)\,\omega_\alpha = DN_\alpha.$$

It is well known ([41, 42]) that ω_α is self adjoint, when $\mathbb{TM}^2$ is endowed with the *First Fundamental Form*, given by

$$I_\alpha(\,.,.\,) = <D\alpha(\,.\,),\, D\alpha(\,.\,)> .$$

Let $\mathcal{K}_\alpha = \det(-\omega_\alpha)$ and $\mathcal{H}_\alpha = \frac{1}{2} trace(-\omega_\alpha)$ be respectively the *Gaussian* and *Mean Curvatures* of the immersion α.

If $(\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha)(p) = 0$, the point $p \in \mathbb{M}^2$ is called an *umbilic point* of α. The (closed) set of umbilic points of α will be denoted by $\mathcal{U}_\alpha$. The eigenvalues $k_{1,\alpha}$, $k_{2,\alpha}$ of $-\omega_\alpha$ are always real and given by

$$k_{1,\alpha} = \mathcal{H}_\alpha - (\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha)^{1/2}, \quad k_{2,\alpha} = \mathcal{H}_\alpha + (\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha)^{1/2}.$$

They are called respectively the *minimal* and *maximal principal curvatures* of α. It holds that $k_{1,\alpha} < k_{2,\alpha}$, except on $\mathcal{U}_\alpha$, where $k_{1,\alpha} = k_{2,\alpha}$.

The *Second Fundamental Form* of α of is defined by

$$II_\alpha(\,.\,,\,.\,) = - <\omega_\alpha(\,.\,),\,.\,> .$$

The *Normal Curvature*, $k_{n,\alpha}(p,l)$, on a line l through the point p, is defined by

$$k_{n,\alpha}(p,l) = \frac{II_\alpha(\,.\,,\,.\,)}{I_\alpha(\,.,.\,)},$$

evaluated at any vector generating the line l.

It is well known ([41, 42]) that at each point $p \in \mathbb{M}^2$, $k_{1,\alpha}(p)$ is the minimum and $k_{2,\alpha}(p)$ is the maximum of $k_{n,\alpha}(p,l)$, taken on all tangent lines l through the point p.

The eigenspaces of $-\omega_\alpha$ associated to the principal curvatures define on $\mathbb{M}^2 \setminus \mathcal{U}_\alpha$ two C^{r-2} line fields $\mathcal{L}_{1,\alpha}$ and $\mathcal{L}_{2,\alpha}$ called respectively the *minimal* and *maximal principal line fields* of α, which are mutually orthogonal in $\mathbb{TM}^2$, with the metric I_α.

The *principal line fields* are characterized by Rodrigues' equations [42, 41]:

$$\mathcal{L}_{i,\alpha} = \{v \in \mathbb{TM}^2;\ \omega_\alpha(v) + k_{i,\alpha} v = 0\} \ \ i = 1,\, 2.$$

Elimination of $k_{i,\alpha},\quad i = 1, 2$ in these equations leads to a single quadratic differential equation $\tau_{g,\alpha} = 0$ for the principal lines of α, in terms of the *geodesic torsion* of α, defined in the direction (.) as

$$\tau_{g,\alpha}(.,.) =< DN_\alpha(.) \wedge D\alpha(.), N_\alpha > .$$

Calculation (see [41, 42]) shows that in a chart (u, v) in $\mathbb{M}^2$, $\tau_{g,\alpha}$ has the form

$$\tau_{g,\alpha}([du : dv]) = \frac{(Fg - Gf)dv^2 + (Eg - Ge)dudv + (Ef - Fe)du^2}{(Edu^2 + 2Fdudv + Gdv^2)\sqrt{EG - F^2}}. \tag{1}$$

The coefficients are functions of (u, v), characterized in terms of the *first* and *second fundamental forms* of the immersion α, written in the chart (u, v) as follows:

$$I_\alpha =< D\alpha, D\alpha >= Edu^2 + 2Fdudv + Gdv^2,$$

with $E =< \alpha_u, \alpha_u >$, $F =< \alpha_u, \alpha_v >$, $G =< \alpha_v, \alpha_v >$, and

$$II_\alpha = - < DN_\alpha,\, D\alpha >=< N_\alpha, D^2\alpha >= edu^2 + 2fdudv + gdv^2,$$

with $e =< N, \alpha_{uu} >= - < N_u, \alpha_u >$, $f =< N, \alpha_{uv} >= - < N_u, \alpha_v >$, $g =< N, \alpha_{vv} >= - < N_v, \alpha_v >$, where $N = N_\alpha$.

The integral curves of $\mathcal{L}_{i,\alpha}\,,\ i = 1, 2$ are called *lines of minimal*, $i = 1$, and *maximal*, $i = 2$, *principal curvature* of α. The family of such curves i.e. the *integral foliation* of $\mathcal{L}_{i,\alpha},\ i = 1, 2$, on $\mathbb{M}^2 \setminus \mathcal{U}_\alpha$ will be denoted by $\mathcal{F}_{i,\alpha},\ i = 1, 2$, and will be called the *minimal*, $i = 1$, and *maximal*, $i = 2$, *principal foliations* of α.

By the *principal configuration of* α will be meant the following triple

$$\mathcal{P}_\alpha = (\mathcal{U}_\alpha, \mathcal{F}_{1,\alpha}, \mathcal{F}_{2,\alpha}).$$

The local study of principal configurations around an umbilic point received considerable attention in the classical works of Monge [33], Cayley [4], Darboux [7] and Gullstrand [23], among others.

More recently Palmeira [34], Guiñez and Gutierrez [20], [19], Guiñez [21], [22], Bruce and Fidal [2] and Bruce and Tari [3], Sánchez-Bringas and Galarza [39], Mello [32], to mention just a few, have studied solutions of more general binary differential equations near singularities.

An initial step to study the generic properties of principal configurations on algebraic surfaces has been given in [17].

The study of the global features of principal configurations $\mathcal{P}_\alpha$ which remain topologically undisturbed under small perturbations of the immersion α –*principal structural stability*– was initiated by Gutierrez and

Sotomayor in [24, 25, 27]. There were established sufficient conditions for immersions α of class C^r of a compact oriented surface $\mathbb{M}^2$ into $\mathbb{R}^3$ to have a C^s-*structurally stable principal configuration*, $r > s \geq 3$. This means that for any immersion β sufficiently C^s-close to α, there must exist a homeomorphism h of $\mathbb{M}^2$ which maps $\mathcal{U}_\alpha$ onto $\mathcal{U}_\beta$ and maps lines of $\mathcal{F}_{i,\alpha},\ i = 1, 2$, to those of $\mathcal{F}_{i,\beta},\ i = 1, 2$.

The conditions for principal structural stability established in [24, 25, 27] are reviewed in sections 3 and 4. They are imposed on each of the topological invariant entities of the principal configuration. Namely,

a) the umbilic points,
b) the periodic lines of curvature, also called principal cycles,
c) the absence of umbilic separatrix connexions, and
d) the absence of non-trivially recurrent principal lines.

For principal structural stability, the umbilic points, which are regarded as the singularities of the principal configuration, are assumed to be *Darbouxian* [7, 24, 27].

In Section 2, the meaning of this assumption is stated intrinsically and in terms of local coordinates involving the coordinate expression for the third order jet of the immersion at an umbilic point. See Fig. 1 or an illustration of the principal configuration around a Darbouxian umbilic. The subscript stands for the number of umbilic separatrices approaching the point. This number is the same for both the minimal and maximal principal curvature foliations.

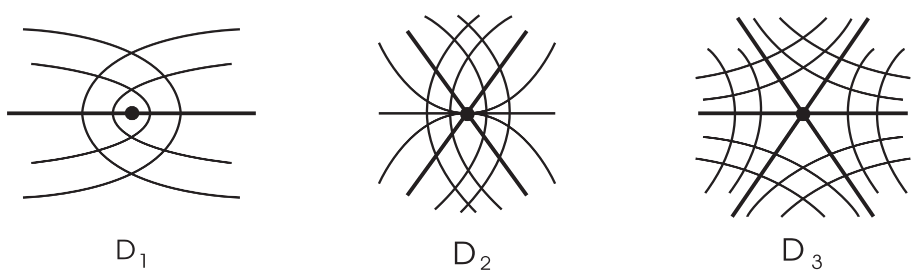

FIGURE 1. Principal curvature lines near the umbilic points D_i and their separatrices

The purpose of this paper is to study the simplest qualitative changes –bifurcations– exhibited by the principal configurations under small perturbations of an immersion which violates in a minimal fashion the Darbouxian structural stability condition on umbilic points.

This leads to two types of umbilic patterns: D_2^1 and $D_{2,3}^1$, illustrated in Fig. 2. The superscript stands for the codimension which is the

minimal number of parameters on which depend the families of immersions exhibiting persistently the pattern. The subscripts stand for the number of separatrices approaching the umbilic. In the first case, this number is the same for both the minimal and maximal principal curvature foliations. In the second case, they are not equal and, in our notation, appear separated by a comma.

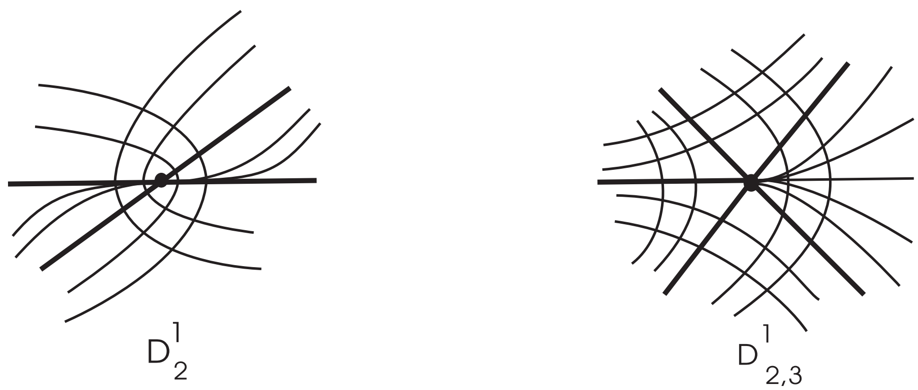

FIGURE 2. Principal curvature lines near the umbilic points D^1_2, left, and $D^1_{2,3}$, right, and their separatrices.

The precise definitions and bifurcation analysis of these points is carried out in a local context in Section 2. The global implications on the bifurcations of principal cycles are studied in sections 3 and 4. This is preceded by a short review of the stability conditions, items *a*), *b*), *c*) and *d*), since they are essential to state properly the above mentioned global implications on the appearance and annihilation of principal cycles as well as for the statements concerning the *relative principal stability* of the immersions at umbilic bifurcations, also included in this paper. This is also done in sections 3 and 4. Section 5 links the results of this paper with other bifurcations of principal configurations discussed in [29].

The study presented here is motivated by the *Theory of First Order Structural Stability* of vector fields on surfaces due to Andronov-Leontovich [1] and further developed by Sotomayor [40]. The conditions *a*), *b*), *c*) and *d*), above, are the counterpart, for the structural stability of principal configurations, of those of Andronov-Pontrjagin [1] and Peixoto [37], for the structural stability of vector fields on surfaces. Particularly meaningful here is the openness and density theorem due to Peixoto [37].

The analogy is pursued here to the case of *first order structural stability*, following the steps of the work of Sotomayor [40], where the

transversality to key Banach submanifolds are instrumental for the geometric explanation of the codimension one generic bifurcation patterns. Elaboration of these ideas gives rise in section 4 to the definition of certain Banach submanifolds consisting of immersions where the simplest umbilic bifurcations happen.

The Theory of Bifurcations of general vector fields has been recently developed in several promising directions. See the books of Chicone [5], and Chow and Hale [6], Guckenheimer and Holmes [18], IIyashenko and Li W. Gu [30] and Roussarie [38], to mention just a few.

The bifurcations of Principal Configurations due to the minimal violation of the structural stability conditions imposed on principal cycles and umbilic separatrix connections have been studied in [26, 28, 16, 29]. To the knowledge of the authors, a systematic analysis of the violation of condition d –leading to non-trivial recurrences of principal lines– has not yet been carried out. Explicit examples of such recurrences – missing in the classical geometry literature – have been provided in [25, 27]. Their classification, however, seems quite distant.

2. Umbilic Points and their Generic Bifurcations

2.1. **Preliminaries concerning umbilic points.** Denote by $\mathbb{PM}^2$ the projective tangent bundle over $\mathbb{M}^2$, with projection Π. For any chart (u, v) on an open set U of $\mathbb{M}^2$ there are defined two charts $(u, v; p = dv/du)$ and $(u, v; q = du/dv)$ which cover $\Pi^{-1}(U)$.

The equation 1 of principal lines, being quadratic, is well defined in the projective bundle. Thus, for every α in $\mathcal{I}^r$,

$$\mathbb{L}_\alpha = \{\tau_{g,\alpha} = 0, \}$$

defines a variety on $\mathbb{PM}^2$, which is regular and of class C^{r-2} over $\mathbb{M}^2 \backslash \mathcal{U}_\alpha$. It doubly covers $\mathbb{M}^2 \backslash \mathcal{U}_\alpha$ and contains a projective line $\Pi^{-1}(p)$ over each point $p \in \mathcal{U}_\alpha$.

Definition 1. A point $p \in \mathcal{U}_\alpha$ is *Darbouxian* if the following two conditions hold:

T : The variety $\mathbb{L}_\alpha$ is regular also over $\Pi^{-1}(p)$. In other words, the derivative of $\tau_{g,\alpha}$ does not vanish on the points of projective line $\Pi^{-1}(p)$. This means that the derivative in directions transversal to $\Pi^{-1}(p)$ must not vanish.

D : The principal line fields $\mathcal{L}_{i,\alpha}$, $i = 1, 2$ lift to a single line field $\mathcal{L}_\alpha$ of class C^{r-3}, tangent to $\mathbb{L}_\alpha$, which extends to a unique one along $\Pi^{-1}(p)$, and there it has only hyperbolic singularities, which must be either

D_1 : a unique saddle

D_2 : a unique node between two saddles, or
D_3 : three saddles.

For calculations it will be essential to express the Darbouxian conditions in a Monge local chart (u, v): $(\mathbb{M}^2, p) \to (\mathbb{R}^2, 0)$ on $\mathbb{M}^2$, $p \in \mathcal{U}_\alpha$, as follows.

Take an isometry Γ of $\mathbb{R}^3$ with $\Gamma(\alpha(p)) = 0$ such that $\Gamma(\alpha(u, v)) = (u, v, h(u, v))$, with

$$\begin{aligned} h(u,v) =& \frac{k}{2}(u^2+v^2) + (a/6)u^3 + (b/2)uv^2 + (b'/2)u^2v \\ &+(c/6)v^3 + (A/24)u^4 + (B/6)u^3v \\ &+(C/4)u^2v^2 + (D/6)uv^3 + (E/24)v^4 + O((u^2+v^2)^{5/2}). \end{aligned} \tag{2}$$

To obtain simpler expressions assume that the coefficient b' vanishes. This is achieved by means of a suitable rotation in the (u, v)-plane.

In the affine chart $(u, v; p = dv/du)$ on $\mathbb{P}(\mathbb{M}^2)$ around $\Pi^{-1}(p)$, the variety $\mathbb{L}_\alpha$ is given by the following equation.

$$\mathcal{T}(u, v, p) = L(u, v)p^2 + M(u, v)p + N(u, v) = 0,\ p = dv/du. \tag{3}$$

According to [41, 42], the functions L, M and N are obtained from equations 1 and 2 as follows:

$$\begin{aligned} L &= h_u h_v h_{vv} - (1 + h_v^2)h_{uv} \\ M &= (1 + h_u^2)h_{vv} - (1 + h_v^2)h_{uu} \\ N &= (1 + h_u^2)h_{uv} - h_u h_v h_{uu}. \end{aligned}$$

Calculation taking into account the coefficients in equation 2, with $b' = 0$, gives:

$$\begin{aligned} L(u,v) =& -bv - B/2(u^2) - (C-k^3)uv - D/2(v^2) + M_1^3(u,v) \\ M(u,v) =& (b-a)u + cv + [(C-A)/2 + k^3]u^2 + (D-B)uv \\ &+[(E-C)/2 - k^3]v^2 + M_2^3(u,v) \\ N(u,v) =& bv + B/2u^2 + (C-k^3)uv + D/2v^2 + M_3^3(u,v), \end{aligned} \tag{4}$$

with $M_i^3(u, v) = O((u^2 + v^2)^{3/2})$, i = 1, 2, 3.

These expressions are obtained from the calculation of the coefficients of the first and second fundamental forms in the chart (u, v) and substitution into 1. See also [7, 24, 27]. With longer calculations, Darboux [7] gives the full expressions for any value of b'.

Remark 1. *The regularity condition T in definition 1 is equivalent to impose that $b(b-a) \neq 0$. In fact, this inequality also implies regularity at $p = \infty$. This can be seen in the chart $(u, v;\, q = du/dv)$, at $q = 0$.*

Also this condition is equivalent to the transversality of the curves $M = 0$, $N = 0$

The line field $\mathcal{L}_\alpha$ is expressed in the chart $(u, v;\, p)$ as being generated by the vector field $X = X_\alpha$, called the *Lie-Cartan* vector field of equation 1, which is tangent to $\mathbb{L}_\alpha$ and is given by:

$$\begin{aligned} \dot{u} =& \partial \mathcal{T}/\partial p \\ \dot{v} =& p \partial \mathcal{T}/\partial p \\ \dot{p} =& -\left(\partial \mathcal{T}/\partial u + p \partial \mathcal{T}/\partial v\right) \end{aligned} \tag{5}$$

Similar expressions hold for the chart $(u, v; q = du/dv)$ and the pertinent vector field $Y = Y_\alpha$.

The function $\mathcal{T}$ is a first integral of $X = X_\alpha$. The projections of the integral curves of X_α by $\Pi(u, v, p) = (u, v)$ are the lines of curvature. The singularities of X_α are given by $(0, 0, p_i)$ where p_i is a root of the equation $p(bp^2 - cp + a - 2b) = 0$.

Assume that $b \neq 0$, which occurs under the regularity condition T, then the singularities of X_α on the surface $\mathbb{L}_\alpha$ are located on the p-axis at the points with coordinates p_0, p_1, p_2

$$\begin{aligned} p_0 =& 0, \\ p_1 =& c/2b - \sqrt{(c/2b)^2 - (a/b) + 2}, \\ p_2 =& c/2b + \sqrt{(c/2b)^2 - (a/b) + 2} \end{aligned} \tag{6}$$

Remark 2. [24] *Assume the notation established in equation 2. Suppose that the transversality condition $T : b(b-a) \neq 0$ of definition 1 and remark 1 holds. Let $\Delta = -[(c/2b)^2 - (a/b) + 2]$. Calculation of the hyperbolicity conditions for singularities 6 of the vector field 5 –see* [24]*– have led to establish the following equivalences:*

$D_1) \equiv \;\; \Delta > 0$

$D_2) \equiv \;\; \Delta < 0 \;\; \textit{and} \;\; 1 < \dfrac{a}{b} \neq 2$

$D_3) \equiv \;\; \dfrac{a}{b} < 1.$

See Figs. 1 and 3 for an illustration of the three possible types of Darbouxian umbilics. The distinction between them is expressed in terms of the coefficients of the 3-jet of equation 2, as well as in the lifting of singularities to the surface $\mathbb{L}_\alpha$. See remarks 1 and 2.

The subscript $i = 1, 2, 3$ of D_i denotes the number of *umbilic separatrices* of p. These are principal lines which tend to the umbilic point p and separate regions of different patterns of approach to it. For Darbouxian points, the umbilic separatrices are the projection into $\mathbb{M}^2$ of the saddle separatrices transversal to the projective line over the umbilic point.

It can be proved that the only umbilic points for which $\alpha \in \mathcal{I}^r$ is locally C^s-structurally stable, $r > s \geq 3$, are the Darbouxian ones. See [2, 27].

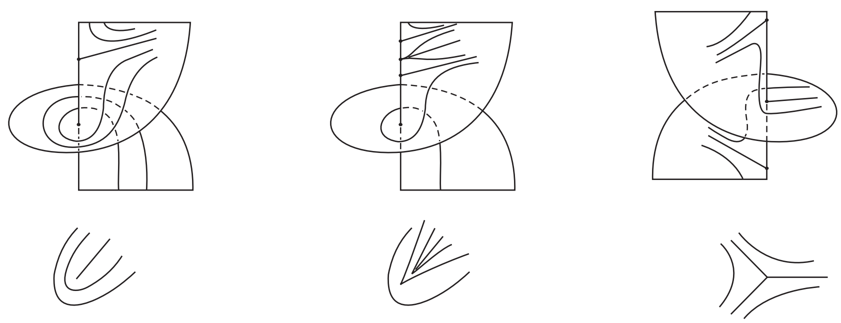

FIGURE 3. Darbouxian Umbilic Points, corresponding $\mathbb{L}_\alpha$ surface and lifted line fields.

The implicit surface $\mathcal{T}(u, v, p) = 0$ is regular in a neighborhood of the projective line if and only if $b(b - a) \neq 0$. Near the singular point $p_0 = (0, 0, 0)$ of X_α it follows that $\mathcal{T}_v(p_0) = b \neq 0$ and therefore, by the Implicit Function Theorem, there exists a function v such that $\mathcal{T}(u, v(u, p), p) = 0$. The function $v = v(u, p)$ has the following Taylor expansion

$$v(u, p) = -\frac{B}{2b}u^2 + \frac{a - b}{b}up + O(3).$$

For future reference we record the expression the vector field X_α in the chart (u, p).

$$\begin{aligned} \dot{u} =& \mathcal{T}_p(u, v(u, p), p) \\ =& (b - a)u + \frac{1}{2}\frac{[b(C - A + 2k^3) - cB]}{b}u^2 + \frac{c(a - b)}{b}up + O(3) \\ \dot{p} =& -(\mathcal{T}_u + p\mathcal{T}_v)(u, v(u, p), p) = -Bu + (a - 2b)p - cp^2 \\ &+ \frac{1}{2}\frac{B(C - k^3) - a_{41}b}{b}u^2 + \frac{b(A - C - 2k^3) + a(k^3 - C)}{b}up + O(3). \end{aligned} \tag{7}$$

where a_{41} is term of order five in h which, however, will have no influence in what follows.

Two generic patterns of bifurcations of umbilic points appear in this work. The first one occurs due to the violation of the Darbouxian condition D, while T is preserved, leading to the pattern called D_2^1, studied in subsection 2.2. The second one happens due to the violation of condition T, leading to the pattern denominated $D_{2,3}^1$, studied in subsection 2.3.

The meaning of the above mentioned genericity assertion for these bifurcations is made precise in subsection 2.4, Theorem 1.

2.2. The D_2^1 Umbilic Bifurcation Pattern.

Here will be studied the qualitative changes - bifurcations - of the principal configurations around non Darbouxian umbilic points such that the regularity (or transversality) condition $T: b(a-b) \neq 0$, which implies their isolatedness, is preserved and only the condition D is violated in the mildest possible way.

Definition 2. A point $p \in \mathcal{U}_\alpha$ is said to be of type D_2^1 if the following holds:

- T : The variety $\mathbb{L}_\alpha$ is regular also over $\Pi^{-1}(p)$. In other words, the derivative of $\tau_{g,\alpha}$ does not vanish on the points of projective line $\Pi^{-1}(p)$. This means that the derivative in directions transversal to $\Pi^{-1}(p)$ must not vanish.
- D_2^1 : The principal line fields $\mathcal{L}_{i,\alpha}$, $i = 1,2$ lift to a single line field $\mathcal{L}_\alpha$ of class C^{r-3}, tangent to $\mathbb{L}_\alpha$, which extends to a unique one along $\Pi^{-1}(p)$, and there it has a hyperbolic saddle singularity and a saddle-node whose central manifold is located along the projective line over p.

In coordinates (u,v), as in the notation above, this means that
$T\ : b(a-b) > 0$ and either
1) $a/b = (c/2b)^2 + 2$, or
2) $a/b = 2$.

We point out that due to the particular representation of the *3-jets* taken here, with $b' = 0$, the space a, b, c in the case 2) is not transversal, but tangent, to the manifold of jets with D_2^1 umbilics.

Remark 3. *The D_2^1 umbilic point has two separatrices.*

The isolated one is characterized by the fact that no other principal line which approaches the umbilic point is tangent to it.

The other separatrix, called non-isolated, has the property that every principal line distinct from the isolated one, that approaches the point does so tangent to it.

These separatrices bound the parabolic sector of lines of curvature approaching the point; they also constitute the boundary of the hyperbolic sector of the umbilic point.

The bifurcation illustrated in Fig. 4 shows that the non-isolated separatrix disappears when the point D_2^1 changes to D_1 and that it turns into an isolated D_2 separatrix when it changes into D_2 . It can be said that D_2^1 represents the simplest transition between D_1 and D_2 Darbouxian umbilic points, which occurs through the annihilation of an umbilic separatrix – the non-isolated one –.

With the notation in equations 3 and 4, write

$$\begin{aligned} L =& -bv + M_1^2(u,v),\\ M =&(b-a)u + cv + M_2^2(u,v),\\ N =& bv + M_3^2(u,v), \end{aligned} \tag{8}$$

with $M_i^2(u,v) = O(u^2+v^2)$.

Condition D_2^1 is equivalent to the existence of a non zero double root for $bp^2 - cp + a - 2b = 0$, which amounts to $b \neq 0$ and $p_1 = p_2 \neq p_0$.

Assuming $b(b-a) \neq 0$, the curves $L = 0$ and $M = 0$ meet transversally at $(0,0)$ if and only if $b \neq a$. It was shown in [24] that D_1 is satisfied if and only if the roots of $bp^2 - cp + a - 2b = 0$ are non vanishing and purely imaginary.

Also, D_2 is satisfied if and only if $bt^2 - ct + a - 2b = 0$ has two distinct non zero real roots, p_1, p_2 which verify $p_1p_2 > -1$.

This means that the rays tangent to the separatrices are pairwise distinct and contained in an open right angular sector.

The local configuration of D_2^1 is established now.

Proposition 1. *Suppose that $\alpha \in \mathcal{I}^r, r \geq 5$, satisfies condition D_2^1 at an umbilic point p. Then the local principal configuration of α around p is that of Fig. 2, right and Fig. 4, center.*

Proof. Consider the Lie-Cartan lifting X_α as in equation 5, which is of class C^{r-3}. If $a = 2b \neq 0$ and $c \neq 0$, it follows that $p_0 = (0,0,0)$ is an isolated singular point of quadratic saddle node type with a center manifold contained in the projective line –the p axis–. In fact, the eigenvalues of $DX_\alpha(0)$ are $\lambda_1 = -b \neq 0$ and $\lambda_2 = 0$ and the p axis is invariant; there X_α, according to equation 7 is given by $\dot{p} = -cp^2 + o(2)$.

The other singular point of X_α is given by $p_1 = (0,0,\frac{c}{b})$. It follows that

$$DX_\alpha(0,0,p_1) = \begin{bmatrix} -b & -c & 0 \\ -c & -\frac{c^2}{b} & 0 \\ A_1 & A_2 & \frac{c^2}{b} \end{bmatrix}$$

where,

$$A_1 = \frac{b^2c(A-k^3-2C)+bc^2(2B-D)+c^3(C-k^3)-b^3D}{b^3}$$
$$A_2 = \frac{b^2c(B-2D)+bc^2(2C+k^3-E)+b^3(k^3-C)+Dc^3}{b^3}$$

The non zero eigenvalues of $DX(0,0,p_1)$ are $\lambda_1 = \frac{c^2}{b}, \quad \lambda_2 = -\frac{c^2+b^2}{b}$. In fact, p_1 is a hyperbolic saddle point of X_α having eigenvalues given by λ_1 and λ_2.

Similar analysis can be done when $(\frac{c}{2b})^2 - \frac{a}{b} + 1 = 0$. In this case X_α and $p_1 = (0,0,\frac{c}{b})$ is a quadratic saddle node, with a local center manifold contained in the projective line. The point $p_0 = (0,0,0)$ is a hyperbolic saddle of X_α. This case is equivalent to the previous one, after a rotation in (u,v) that sends de saddle-node to $p = 0$.

□

The principal configuration of the D_2^1 umbilic point is illustrated in Fig. 2, left, and its bifurcation in Fig. 4.

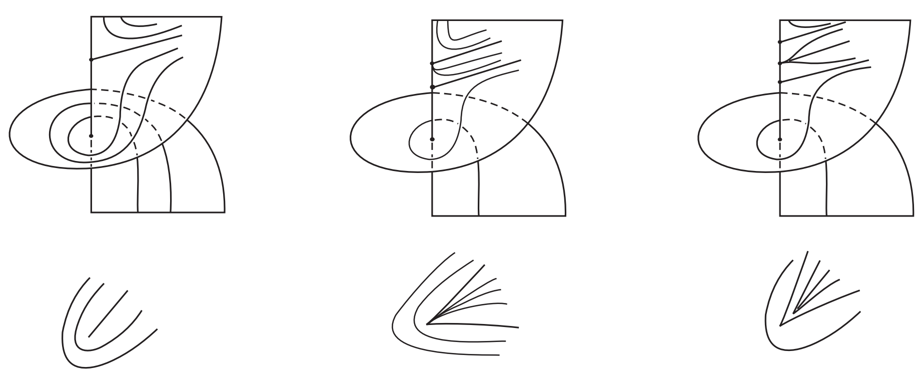

FIGURE 4. Umbilic Point D_2^1 , corresponding $\mathbb{L}_\alpha$ surface and lifted line fields.

Proposition 2. *Suppose that $\alpha \in \mathcal{I}^r, r \geq 5$, satisfies condition D_2^1 at an umbilic point p. Then there is a function $\mathcal{B}$ of class C^{r-3} on a neighborhood $\mathcal{V}$ of α and a neighborhood V of p such that every $\beta \in \mathcal{V}$ has a unique umbilic point p_β in V.*

i) $d\mathcal{B}(\alpha) \neq 0$
ii) $\mathcal{B}(\beta) > 0$ *if and only if* p_β *is Darbouxian of type* D_1
iii) $\mathcal{B}(\beta) < 0$ *if and only if* p_β *is Darbouxian of type* D_2
iv) $\mathcal{B}(\beta) = 0$ *if and only if* p_β *is of type* D_2^1

The principal configurations of β *around* p *is that of Fig. 4, left, right and center, respectively.*

Proof. Since p is a transversal umbilic point of α, the existence of the neighborhoods $\mathcal{V}$ and V of p_β follow from the Implicit Function Theorem. So we assume that after an isometry Γ_β of $\mathbb{R}^3$, with $\Gamma_\beta\beta(0) = 0$, in the neighborhood V are defined coordinates (u, v), also depending on β, on which it is represented as:

$$h_\beta(u,v) = k_\beta/2(u^2+v^2)+a_\beta/6(u^3)+b_\beta/2(uv^2)+c_\beta/6(v^3)+O(\beta;(u^2+v^2)^4).$$

Define the function

$$\mathcal{B}(\beta) = [c_\beta/2b_\beta]^2 - a_\beta/b_\beta + 2,$$

whose zeros define locally the manifold of immersions with a D_2^1 point.

The derivative of this function in the direction of the coordinate a is clearly non-zero. □

2.3. **The $D^1_{2,3}$ Umbilic Bifurcation Pattern.** The second case of non-Darbouxian umbilic point studied here, called $D^1_{2,3}$, happens when the regularity condition T is violated.

Definition 3. An umbilic point is said of type $D^1_{2,3}$ if the transversality condition T fails at two points over the umbilic point, at which $\mathbb{L}_\alpha$ is non-degenerate of *Morse* type.

Proposition 3. *Suppose that* $\alpha \in \mathcal{I}^r$, $r \geq 5$, *and* p *be an umbilic point. Assume the notation in 2 with* $b = a \neq 0$ *and* $b(C - A + 2k^3) - cB \neq 0$.

Then p *is of type* $D^1_{2,3}$ *and the local principal configuration of* α *around* p *is that of Fig. 2, right.*

Proof. Consider the Lie-Cartan lifting X_α as in equation 5, which is of class C^{r-3}. Imposing $a = b \neq 0$, by equations 6 and 7, the singular points of X_α are p_0, p_1 and p_2, roots of the equation $p(bp^2 - cp - b) = 0$.

In fact, if $a = b \neq 0$, it follows that p_0 is a quadratic saddle node with center manifold transversal to the projective line.

From equation 7 the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -b$ and the all center manifolds W^c are tangent to the line $p = -\frac{B}{b}u$. By invariant

manifold theory it follows that $X|W^c$ is local topologically equivalent to

$$\dot{u} = \frac{1}{2}\frac{[b(C - A + 2k^3) - cB]}{b}u^2 + o(2) := -\frac{\chi}{2b}u^2 + o(2). \qquad (9)$$

It follows that

$$DX_\alpha(0,0,p_i) = \begin{bmatrix} 0 & -2bp_i + c & 0 \\ 0 & -p_i(2bp_i - c) & 0 \\ B_1 & B_2 & 3bp_i^2 - 2cp_i - b \end{bmatrix}$$

where,

$$\begin{aligned} B_1 =& (C - k^3)p_i^3 + (2B - D)p_i^2 + (A - 2C - k^3)p_i - B \\ B_2 =& Dp_i^3 + (2C + k^3 - E)p_i^2 + (B - 2D)p_i + k^3 - C. \end{aligned}$$

The nonzero eigenvalues of $DX_\alpha(0,0,p_i)$ are $\lambda_1 = -2bp_i^2 + cp_i = -b(p_i^2 + 1)$ and $\lambda_2 = 3bp_i^2 - 2cp_i - b = b(p_i^2 + 1)$.

By invariant manifold theory, $(0,0,p_i)$ are saddles of X_α. The phase portrait of X_α near these singularities are as shown in Fig. 5.

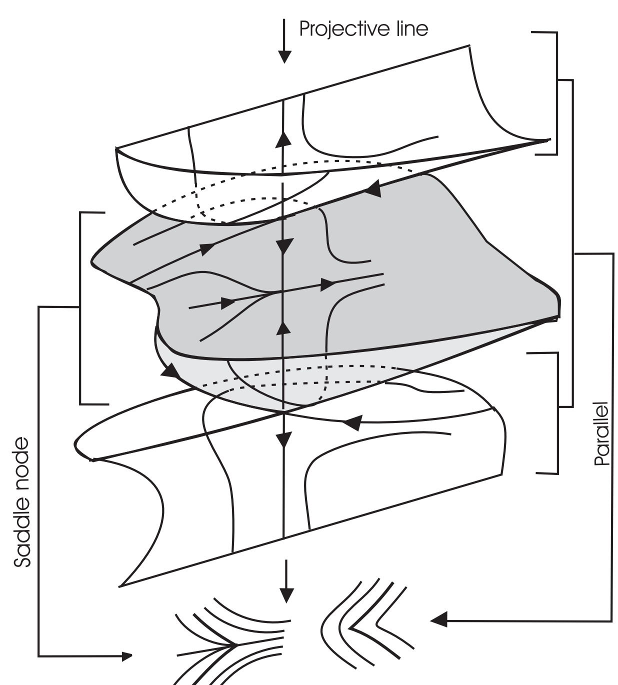

FIGURE 5. Lie-Cartan suspension $D^1_{2,3}$

The two *critical* points p_1 and p_2 are of conic type on the variety $\mathbb{L}_\alpha$ over the umbilic point.

These points are non-degenerate or of *Morse* type, according to the analysis below. At the points $(0,0,p_i)$ the variety $\mathcal{T}(u,v,p)=0$ is not regular. In fact:

$\nabla\mathcal{T}(0,0,p)=[(b-a)p,-bp^2+cp+b,0]$.

Therefore, for $a=b\neq 0$, at the two roots of the equation $-bp^2+cp+b=0$ it follows that $\nabla\mathcal{T}(0,0,p_i)=(0,0,0), i=1,2$.

The Hessian of $\mathcal{T}$ at $(0,0,p_i)$ is

$$Hess(\mathcal{T})(0,0,p_i)=\begin{bmatrix} \frac{p_i(-cB+b(C+2k^3-A))}{b} & \frac{p_i(c(k^3-C)+b(D-B))}{b} & 0 \\ \frac{p_i(c(k^3-C)+b(D-B)}{b} & -\frac{p_i(cD+b(C-E+2k^3))}{b} & -2bp_i+c \\ 0 & -2bp_i+c & 0 \end{bmatrix}$$

Direct calculation with the notation in equation 9 gives,

$$det(Hess(\mathcal{T})(0,0,p_i))=\frac{p_i(-2bp_i+c)^2\chi}{b}=\frac{b}{p_i}(p_i^2+1)^2\chi\neq 0.$$

Therefore, $(0,0,p_i)$ is a non degenerate critical point of $\mathcal{T}$ of Morse type and index 1 or 2 –a *cone*–, since $\mathcal{T}^{-1}(0)$ contains the projective line. □

Remark 4. *Our analysis has shown the equivalence between the conditions* a) *and* b) *that follow:*

a) The non-vanishing on the Hessian of $\mathcal{T}$ on the critical points p_1 and p_2 over the umbilic.

b) The presence of a saddle-node at p_0 on the regular portion of the surface $\mathbb{L}_\alpha$, with central separatrix transversal to the projective line over the umbilic.

Further direct calculation with equation 4 gives that these two conditions are equivalent to

c) The quadratic contact at the umbilic between the curves $M=0$ and $N=0$.

In fact, from equation 4 it follows that $M(u,v(u))=0$ for $v=-(B/2b)u^2+o(2)$ of class C^{r-2}. Therefore $n(u)=N(u,v(u))$ is of class C^{r-2} and $n(u)=-(\chi/2b)u^2+o(2)$.

Notice also that, unlike the other umbilic points discussed here, the two principal foliations around $D^1_{2,3}$ are topologically distinct.

One of them, located on the *parallel sheet*, has two umbilic separatrices and two hyperbolic sectors

The other, located on the *saddle-node sheet*, has three umbilic separatrices, one parabolic and two hyperbolic sectors.

The separatrix which is the common boundary of the hyperbolic sectors will be called *hyperbolic separatrix*. See Figs. 3, 4 and Fig. 5 for illustrations.

The bifurcation analysis describes the elimination of two umbilic points D_2 and D_3 which, under a deformation of the immersion, collapse into a single umbilic point $D^1_{2,3}$, and then, after a further suitable arbitrarily small perturbation, the umbilic point is annihilated.

Proposition 4. *Suppose that $\alpha \in \mathcal{I}^r, r \geq 5$, satisfies condition $D^1_{2,3}$ at an umbilic point p. Then there is a function $\mathcal{B}$ of class C^{r-3} on a neighborhood $\mathcal{V}$ of α and a neighborhood V of p such that*

i) $d\mathcal{B}(\alpha) \neq 0$

ii) $\mathcal{B}(\beta) > 0$ *if and only if β has no umbilic points in V,*

iii) $\mathcal{B}(\beta) < 0$ *if and only if β has two Darbouxian umbilic points of types D_2 and D_3,*

iv) $\mathcal{B}(\beta) = 0$ *if and only if β has only one umbilic point in V, which is of type $D^1_{2,3}$.*

The principal configurations of β around p are illustrated in Fig. 6, right, left and center, respectively.

Proof. Similar to that given in [40], page 15, for the saddle-node of vector fields, using the equivalence c) of remark 4. We define $\mathcal{B}$ as follows. An immersion β in a neighborhood $\mathcal{V}$ of α and a neighborhood V of p can be written in a Monge chart as a graph of a function $h_\beta(u, v)$. The umbilic points of β are defined by the equation

$$\begin{aligned} M_\beta &= (1 + ((h_\beta)_u)^2)(h_\beta)_{vv} - (1 + ((h_\beta)_v)^2)(h_\beta)_{uu} = 0 \\ N_\beta &= (1 + ((h_\beta)_u)^2)(h_\beta)_{uv} - (h_\beta)_u(h_\beta)_v(h_\beta)_{uu} = 0. \end{aligned} \tag{10}$$

For β in a neighborhood of α it follows that $M_\beta(u, v_\beta(u)) = 0$.

Define $\mathcal{B}(\beta) = n_\beta(u_\beta)$, where u_β is the only critical point of $n_\beta(u) = N_\beta(u, v_\beta(u))$.

Taking $h_\beta(u, v) = h(u, v) + \lambda uv$, where h is as in equation 2 it follows by direct calculation that $\frac{d\mathcal{B}(\beta)}{d\lambda}|_{\lambda=0} \neq 0$. □

The bifurcation of the point $D^1_{2,3}$ can be regarded as the simplest transition between umbilics D_2 and D_3 and non umbilic points. See the illustration in Fig. 6.

2.4. **A Transversality Theorem.**

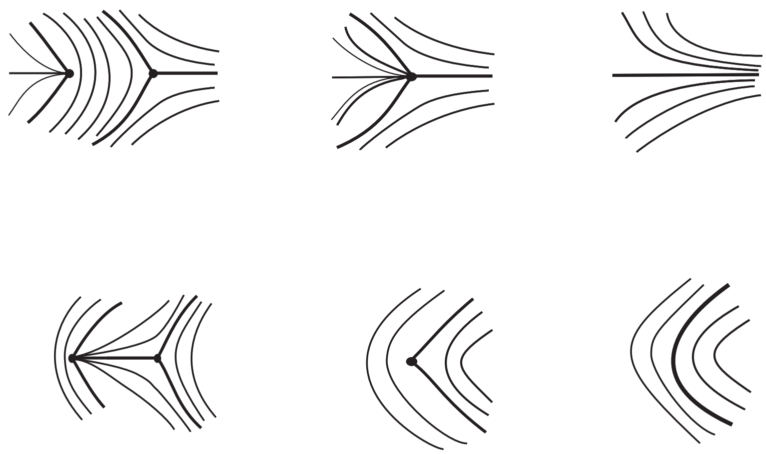

FIGURE 6. Umbilic Point $D^1_{2,3}$ and bifurcation.

Theorem 1. In the space of smooth mappings of $\mathbb{M}^2 \times \mathbb{R} \to \mathbb{R}^3$ which are immersions relative to the first variable, those which have their umbilic points Darbouxian, D^1_2 and $D^1_{2,3}$, forming a curve in $\mathbb{M}^2 \times \mathbb{R}$ whose projection into $\mathbb{R}$ has only non-degenerate critical points at $D^1_{2,3}$, is open and dense.

Proof. Follows from Thom Transversality Theorem applied to the submanifold of three jets of immersions at umbilic points, stratified by the Darbouxian, having codimension 2, D^1_2 and $D^1_{2,3}$, having codimension 3, and their complement having codimension larger than or equal to 4.

Figures 7 and 8 illustrate the unfolding of umbilic points in a generic one parameter family of immersions. $\square$

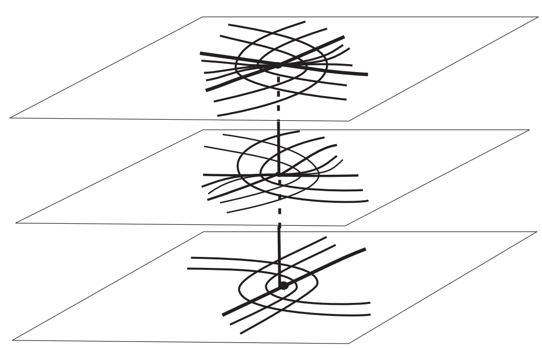

FIGURE 7. Bifurcation of D^1_2

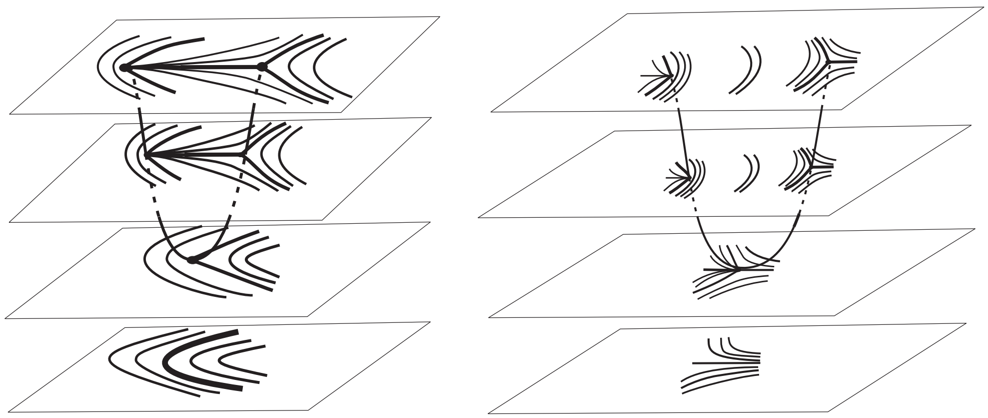

FIGURE 8. Bifurcation of $D^1_{2,3}$

Remark 5. *Although pictures equivalent to Fig. 8 – $D^1_{2,3}$— appear in previous printed work, no proofs seem to have been provided for them before. See Gullstrand* [23] *and Porteous* [36].

Remark 6. *There is a close relationship between one-parameter families of immersed surfaces in $\mathbb{R}^3$ and immersions of 3-manifolds in $\mathbb{R}^4$.*

Assume that an umbilic point p at the value 0 of the parameter λ in a family α_λ is non flat –$k \neq 0$–. By means of an inversion, which does not modify the principal configuration, this can be achieved.

The principal configuration of the family around $(p, 0)$ is the same as that of the immersion $(u, v, \lambda) \to (\alpha_\lambda(u, v), \lambda)$.

The generic bifurcation patterns D^1_2 and $D^1_{2,3}$ studied here correspond to the generic partially umbilic lines for immersions of 3-manifolds in $\mathbb{R}^4$. See Garcia [8], [9].

3. Global Implications of Umbilic Bifurcations

The local bifurcations at umbilic points discussed above have global implications on the principal configurations of the immersions. In order to formulate precisely the pertinent results it is necessary to review some terminology established in previous papers. See [24, 25, 27]

3.1. **Principal Cycles.** A compact line c of $\mathcal{L}_{1\alpha}$ (resp. $\mathcal{L}_{2\alpha}$) is called minimal (resp. maximal) principal cycle of α.

Call $\pi = \pi_c$ the Poincaré first return map (holonomy) defined by the lines of the foliation to which c belongs, defined on a segment of a line of the orthogonal foliation through a point o in c. A cycle is called hyperbolic if $\pi'(0) \neq 1$. It has been proved in [24] that c is hyperbolic if and only if one of the following, equivalent conditions hold

a) $\int_c dk_{2,\alpha}/(k_{2,\alpha} - k_{1,\alpha}) = \int_c dk_{1,\alpha}/(k_{2,\alpha} - k_{1,\alpha}) \neq 0$

b) $\int_c \frac{d\mathcal{H}_\alpha}{\sqrt{\mathcal{H}_\alpha^2 - \mathcal{K}_\alpha}} \neq 0.$

The simplest bifurcation of non-hyperbolic principal cycles has been studied in [27] and [16].

3.2. **Umbilic Connections and Loops.**

A principal line γ which is an *umbilic separatrix* of two different umbilic points p, q of α or twice a separatrix of the same umbilic point p of α is called an *umbilic separatrix connection* of α; in the second case γ is also called an *umbilic separatrix loop.* The simplest bifurcations of umbilic connections as well as the consequent appearance of principal cycles have been studied in [28].

There are two bifurcation patterns producing principal cycles which will be studied in this work. They are associated with the bifurcations of D_2^1 and $D_{2,3}^1$ umbilic points, when their separatrices form loops, self connecting these points. They are defined as follows.

A D_2^1 - *interior loop* consists on a point of type D_2^1 and its isolated separatrix, which is assumed to be contained in the interior of the parabolic sector. See Fig. 9 , where such loop together with the bifurcating principal cycle are illustrated. Here, a hyperbolic principal cycle bifurcates from the loop when the D_2^1 point bifurcates into D_1 .

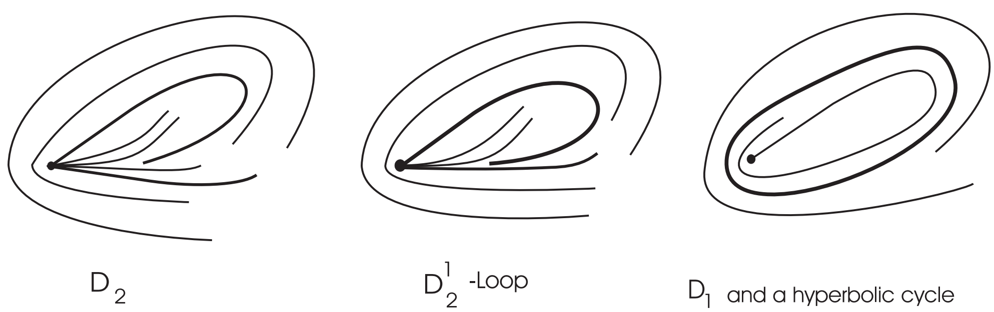

FIGURE 9. D_2^1 - loop bifurcation

If both principal foliations have D_2^1 - interior loops (at the same D_2^1 point), after bifurcation there appear two hyperbolic cycles, one for each foliation. This case will be called *double* D_2^1 - *interior loop.* In Fig. 10, Fig. 9 has been modified and completed accordingly so as to represent both maximal and minimal foliations, each with its respective D_2^1 -interior loops (left) and bifurcating hyperbolic principal cycles (right).

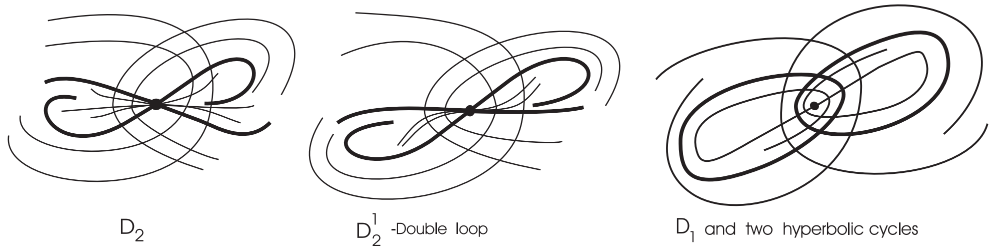

FIGURE 10. D_2^1 - double loop bifurcation

Proposition 5. *A unique hyperbolic principal cycle bifurcates from a D_2^1 loop.*

Proof. The proof follows from the same analysis leading to the uniqueness and hyperbolicity of the periodic orbit bifurcating from the singular cycle consisting of separatrix connecting a saddle-node and a saddle through a separatrix located at projective line and a finite saddle separatrix, interior to the parabolic sector of the saddle-node. See [35].

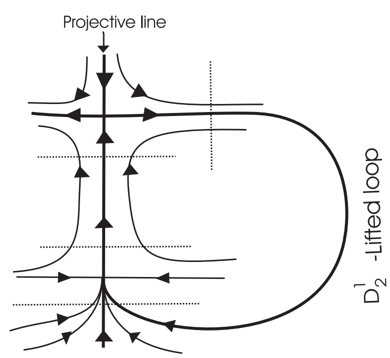

FIGURE 11. D_2^1 - lifted loop.

This consists in the decomposition of the return map into a singular transition (across the saddle-node point), whose contracting behavior is very small, even when compared with the transition along a saddle hyperbolic cycle, and two regular transitions along the separatrix at infinity and a regular one (along the finite saddle separatrix), whose expansion behavior is bounded. See Fig. 11. □

A $D_{2,3}^1$ - *interior loop* consists on a point of type $D_{2,3}^1$ and its hyperbolic separatrix, which is assumed to be contained in the interior of the parabolic sector. See Fig. 12, where such loop together with the

bifurcating principal cycle are illustrated. Here, a hyperbolic principal cycle bifurcates from the loop when the umbilic points are annihilated.

Notice that the D_2^1 and $D_{2,3}^1$ interior loops described above, being separatrices at only one and, are neither umbilic separatrix connections or loops in the sense of Gutierrez and Sotomayor [28].

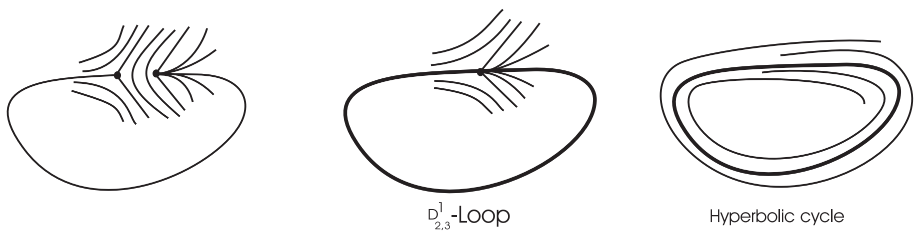

FIGURE 12. $D_{2,3}^1$ - loop bifurcation.

Proposition 6. *A unique hyperbolic principal cycle bifurcates from a $D_{2,3}^1$ loop.*

Proof. The proof follows from the same analysis leading to the uniqueness and hyperbolicity of the saddle-node loop bifurcation of vector fields on 2-manifolds [40]. This consists in the decomposition of the return map into a singular transition (across the saddle-node point), whose contracting behavior is very small and a regular one (along the loop), whose expansion behavior is bounded. □

4. Relative Principal Structural Stability and Globalization.

Let $\mathcal{C}$ be a subset of $\mathcal{I}^{r,s}$. An element $\alpha \in \mathcal{C} \subset \mathcal{I}^{r,s}$ is said C^s- structurally stable relative to $\mathcal{C}$, if there is a neighborhood $\mathcal{V}$ of α in $\mathcal{I}^{r,s}$ such that for every $\beta \in \mathcal{C} \cap \mathcal{V}$ there is a homeomorphism $h = h_\beta$ of $\mathbb{M}^2$ which maps $\mathcal{U}_\alpha$ onto $\mathcal{U}_\beta$ and maps lines of $\mathcal{F}_{i,\alpha}$ onto those of $\mathcal{F}_{i,\beta}$, $i = 1, 2$.

When $\mathcal{C}$ is the whole $\mathcal{I}^{r,s}$, α is called simply C^s-structurally stable.

Call $\mathcal{S}^r(j)$, $j = a, b, c, d$, the set of $\alpha \in \mathcal{I}^r$, $r \geq 4$ such that, respectively,

a) all the umbilic points of α are Darbouxian,
b) all the principal cycles of α are hyperbolic,
c) α has no umbilic separatrix connections,
d) the limit set of every principal line of α is the union of umbilic points, principal cycles and umbilic connections.

The basic stability and genericity result that follows provides a synthesis of these conditions.

Theorem 2. ([24, 25, 27]). Let $r \geq 4$. The set $\mathcal{S}^r = \cap_{\{j=a,\cdots,d\}}\mathcal{S}^r(j)$, is open in $\mathcal{I}^{r,3}$ and dense in $\mathcal{I}^{r,2}$. Every $\alpha \in \mathcal{S}^r$ is C^3-structurally stable.

The structural stability for lines of curvature has been extended recently to other equations and foliations of classical geometry. See [10], [11], [12], [13], [14], [15],

Recall that a *Banach submanifold* of class C^k and codimension one of the Banach manifold $\mathcal{I}^{r,r} = \mathcal{I}^{r,r}(\mathbb{M}^2, \mathbb{R}^3)$ is a subset $\mathcal{B}$ locally implicitly defined by the set of zeroes of a real valued C^k function with non vanishing derivative, see [40] and [31].

Now it is possible to state the main global results of this paper.

Call $\mathcal{S}^r(a_1)$ the set of $\alpha \in \mathcal{I}^r \backslash \mathcal{S}^r(a)$ with a non-Darbouxian umbilic point at which the transversality condition T holds.

Theorem 3. Let $r \geq 5$. Then the following holds.

i) The set $\mathcal{S}_1^r(a_1)$ of immersions $\alpha \in \cap_{\{j=b,c,d\}}\mathcal{S}^r(j)$, such that all their umbilic points are Darbouxian except one which is of type D_2^1, is a Banach submanifold of codimension 1 and of class C^{r-3} of $\mathcal{I}^{r,r}$.

ii) The set $\mathcal{S}_1^r(a_1)$ is open in $\mathcal{I}^{r,4} = \mathcal{I}^r \backslash \mathcal{S}^r$ endowed with the C^3-topology, and is dense in $\mathcal{I}_1^{r,2}(a_1) = \mathcal{I}_1^r(a_1)$ endowed with the C^2-topology.

iii) Every $\alpha \in \mathcal{S}_1^r(a_1)$ is C^4-structurally stable relative to $\mathcal{S}^r(a_1)$.

Proof. Outline. Through the Lie-Cartan lifting, the proof is localized and reduced to the case of vector fields where the Banach structure established in the work of Sotomayor [40] applies. In this case the center manifold of the saddle-node is tangent to the projective line and the characterization of the umbilic point D_2^1 is in terms of the 3-jet of the immersion. The methods of canonical regions and their continuous dependence of the immersions used in [24] apply with minor modifications to the present case to achieve the relative openness and construction of homomorphism preserving principal configurations between immersions inside $\mathcal{S}_1^r(a_1)$ which are close to each other. The approximation leading to the $C^{r,2}$ density is similar to that developed by Gutierrez and Sotomayor [25, 27] to prove the $C^{r,2}$ density of $\mathcal{S}^r$. □

Call $\mathcal{I}_1^r(a_2)$ the set of $\alpha \in \mathcal{I}^r \backslash \mathcal{S}^r(a)$ with an umbilic point at which the transversality condition T does not hold.

Theorem 4. Let $r \geq 5$. Then the following holds

i) The set $\mathcal{S}_1^r(a_2)$ of immersions $\alpha \in \cap_{\{j=b,c,d\}} \mathcal{S}^r(j)$, such that all their umbilic points are Darbouxian except one which is of type $D^1_{2,3}$, is a Banach submanifold of codimension 1 and of class C^{r-3} of $\mathcal{I}^{r,r}$.

ii) The set $\mathcal{S}_1^r(a_2)$ is open in $\mathcal{I}^{r,5} = \mathcal{I}^r \backslash \mathcal{S}^r$ endowed with the C^3-topology, and is dense in $\mathcal{I}_1^{r,2}(a_2) = \mathcal{I}_1^r(a_2)$ endowed with the C^2-topology.

iii) Every $\alpha \in \mathcal{S}_1^r(a_2)$ is C^5-structurally stable relative to $\mathcal{S}^r(a_2)$.

Proof. Outline. Through the Lie-Cartan lifting, the proof is localized and reduced to the case of vector fields where the Banach structure established in the work of Sotomayor [40] applies. In this case the center manifold of the saddle-node is transversal to the projective line and the characterization of the umbilic point $D^1_{2,3}$ is in terms of the 4-jet of the immersion. The methods of canonical regions and their continuous dependence of the immersions used in [24] apply with minor modifications to the present case to achieve the relative openness and construction of homomorphism preserving principal configurations between immersions inside $\mathcal{S}_1^r(a_2)$ which are close to each other. The approximation leading to the $C^{r,2}$ density is similar to that developed by Gutierrez and Sotomayor [25, 27] to prove the $C^{r,2}$ density of $\mathcal{S}^r$. □

5. Concluding Remark

In this work we have established that Figures 4 or 7 and 6 or 8, completed when pertinent with the global implications in Figs. 9, 10 and 12, represent the topological changes on the principal configurations of families of immersions depending on a parameter. These bifurcations are exhibited when the families cross transversally the codimension one submanifolds $\mathcal{S}_1^r(a_1)$ and $\mathcal{S}_1^r(a_2)$. This contribution, together with those presented in [26] and [28], completes the analysis of codimension one bifurcations for principal configurations formulated in [29].

References

[1] Andronov A., Leontovich E. et al. , Theory of bifurcations of dynamical systems on a plane, ITPS, Jerusalem, 1971.

[2] Bruce, J. W. , Fidal D. L., On binary differential equations and umbilics, Proceedings of The Royal Soc. of Edinburgh, 111A, (1987), 147–168.

[3] Bruce, J. W. , Tari F., On binary differential equations, Nonlinearity 8, (1995) 255–271.

[4] Cayley A., On differential equations and umbilici. Philos. Mag. 26, 1863, Collected Works, Vol. vi.

[5] Chicone C. Ordinary differential equations and applications, Texts in Applied Mathematics, **34**, Springer Verlag, 1999.

[6] Chow S. and Hale J., Methods of bifurcation theory, Springer Verlag, 1982.

[7] Darboux G., Sur la forme des lignes de courbure dans la voisingage d'un ombilic. Leçons sur la théorie des surfaces, IV, Note 7, Gauthier Villars, 1896.

[8] Garcia R. , Lines of curvature near partially umbilic points of hypersurfaces of $\mathbb{R}^4$, Prépublication do Laboratoire de Topologie, Université de Bourgogne, **27**, (1993), 1-36.

[9] Garcia R. , Principal curvature lines near Darbouxian partially umbilic points of hypersurfaces immersed in $\mathbb{R}^4$, Computational and Applied Mathematics, SBMAC, **20**, (2001), 121-148.

[10] Garcia R. , Gutierrez C., Sotomayor J., Structural stability of asymptotic lines on surfaces immersed in $\mathbb{R}^3$. Bull. Sci. Math. **123** (1999), no. 8, 599–622.

[11] Garcia R. , Mello L.F., Sotomayor J., Principal Mean Curvature Foliations on Surfaces immersed in $\mathbb{R}^4$, to appear in Proceedings Equadiff-2003, math.DS/0311215, www.arxiv.org.

[12] Garcia R. , Sotomayor J., Lines of axial curvature on surfaces immersed in $\mathbb{R}^4$. Differential Geom. Appl. **12** (2000), 253–269.

[13] Garcia R. , Sotomayor J., Structurally stable configurations of lines of mean curvature and umbilic points on surfaces immersed in $\mathbb{R}^3$. Publ. Mat. **45** (2001), 431–466.

[14] Garcia R. , Sotomayor J., Geometric mean curvature lines on surfaces immersed in $\mathbb{R}^3$. Annales de la Faculté de Sciences de Toulouse, **11**, (2002), 377-401.

[15] Garcia R. , Sotomayor J., Harmonic mean curvature lines on surfaces immersed in $\mathbb{R}^3$. Bull. Braz. Math. Soc. **34** (2003), 303–331.

[16] Garcia R. , Sotomayor J., Principal lines near principal cycles, Annals of Global Analysis and Geometry, **10**, (1992), 275-289.

[17] Garcia R., Sotomayor J., Lines of curvature on algebraic surfaces. Bull. des Sciences Math. **120**, (1996), 367–395.

[18] Guckenheimer J. , Holmes P. Nonlinear oscillations, dynamical systems and bifurcations of vector fields, Applied Math. Sciences, **42**, Second edition, Springer Verlag, 1986.

[19] Guiñez V., Gutierrez C., Positive Quadratic Differential Forms: Linearization, Finite Determinacy and Versal Unfolding. Annales de Toulouse, Série 6, Volume V, Fasc. 4, (1996), 661-690.

[20] Guiñez V., Gutierrez C., Rank-1 codimension one singularities of positive quadratic differential forms, Cadernos de Matemática, ICMC-USP, **04**, (2003), 87-114.

[21] Guiñez V., Rank two codimension 1 singularities of positive quadratic differential forms, Nonlinearity **10**, (1997), 631–654.

[22] Guiñez V., Positive quadratic differential forms and foliations with singularities on surfaces, Tans. Amer. Math. Soc., **309**, (1988) 477–502.

[23] Gullstrand A., Zur Kenntiss der Kreispunkte, Acta Math, **29**, 1905.

[24] Gutierrez C., Sotomayor J., Stable configurations of lines of principal curvature, Asterisque, **98-99**, (1982), 195–215

[25] Gutierrez C. , Sotomayor J., An approximation theorem for immersions with stable configurations of lines of principal curvature, Springer Lect. Notes in Math., **1007**, (1983), 332–368

[26] Gutierrez C., Sotomayor J., Closed principal lines and bifurcations, Bol Soc. Bras. Mat, **17**, (1986), 1–19.

[27] Gutierrez C., Sotomayor J. , Lines of curvature and umbilic points on surfaces, Lecture Notes, 18^{th} Brazilian Math. Colloq , IMPA, 1991. Reprinted and updated as Structurally Stable Configurations of Lines of Curvature and Umbilic Points on Surfaces, Lima, Monografias del IMCA, 1998.

[28] Gutierrez C., Sotomayor J., Periodic lines of curvature bifurcating from Darbouxian umbilical connections, Springer Lect. Notes in Math., **1455**, (1990), 196–229.

[29] Gutierrez C. , Sotomayor J. , Bifurcations of lines of curvature and umbilic points, Aportaciones Matematicas, Soc. Mat. Mex. **01**, (1985), 115–126.

[30] Ilyashenko Yu., Li Wei Gu, Nonlocal Bifurcations, Surveys and Monographs, **66**, American Math. Soc., 1999.

[31] Lang S., Introduction to differentiable manifolds, Springer Verlag, 2002.

[32] Mello, L. F., Mean Directionally Curved Lines on Surfaces Immersed in $\mathbb{R}^4$. Publicacions Matematiques, **47:2**, (2003), 415–440.

[33] Monge G., Sur les lignes de courbure de la surface de l'ellipsoide, Journ. Ecole Polytech., II cah. 1796.

[34] Palmeira C. F., Line fields defined by eigenspaces o derivatives of maps from the plane into itself, Curs. y Congr., 61, Univ. Santiago de Compostela, Spain, 1989.

[35] Paterlini R., Sotomayor J., Bifurcations of polynomial vector fields, Oscillations, Bifurcation and Chaos (Toronto, Ont. 1986), 665-685, CMS Conf. Proc., 8, Amer. Math. Soc., Providence, RI, 1987.

[36] Porteous I. R., Geometric Differentiation, Cambridge University Press, 1994.

[37] Peixoto M., Structural stability on two-dimensional manifolds, Topology, **01**, (1962), 101–120.

[38] Roussarie R., Bifurcations of Planar Vector Fields and Hilbert's Sixteen Problem, Progress in Mathematics, **164**, Birkhaüser Verlag, Basel, 1998.

[39] Sánchez-Bringas F., Ramírez-Galarza A., Lines of curvature near umbilical points on surfaces immersed in $\mathbb{R}^4$. Ann. Global Anal. Geom. **13** (1995), 129–140.

[40] Sotomayor J., Generic one paramater families of vector fields on two dimensional manifolds, Publ. Math. IHES, **43**, (1974), 5–46.

[41] Spivak M., Introduction to Comprehensive Differential Geometry, Vol. III, IV Berkeley, Publish or Perish, 1980.

[42] Struik D., Lectures on classical differential geometry, Addison Wesley, 1950. Reprinted by Dover Publications, Inc., 1988.

[43] Thom R., Stabilité Structurelle et Morphogenese, Benjamin, 1972.

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